

# Identifying Complexity by Means of Matrices

S. Drożdż<sup>1,2,3,4</sup>, J. Kwapień<sup>1,2</sup>, J. Speth<sup>1</sup>, M. Wójcik<sup>2</sup>

<sup>1</sup>*Institut für Kernphysik, Forschungszentrum Jülich, D-52425 Jülich, Germany,*

<sup>2</sup>*Institute of Nuclear Physics, PL-31-342 Kraków, Poland,*

<sup>3</sup>*Institute of Physics, University of Rzeszów, PL-35-310 Rzeszów, Poland,*

<sup>4</sup>*Physikalisches Institut, Universität Bonn, D-53115 Bonn, Germany*

*Dedicated to Professor Dr. H.E. Stanley on the occasion of his 60th birthday*

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## Abstract

Complexity is an interdisciplinary concept which, first of all, addresses the question of how order emerges out of randomness. For many reasons matrices provide a very practical and powerful tool in approaching and quantifying the related characteristics. Based on several natural complex dynamical systems, like the strongly interacting quantum many-body systems, the human brain and the financial markets, by relating empirical observations to the random matrix theory and quantifying deviations in term of a reduced dimensionality, we present arguments in favour of the statement that complexity is a phenomenon at the edge between collectivity and chaos.

*Key words:* Natural complex systems, Random matrix theory, Order out of randomness

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## 1 Introduction

By its very nature, even though central to the contemporary physics, the concept of complexity still lacks a precise definition. In qualitative terms this concept refers to diversity of forms, to emergence of coherent patterns out of randomness and also to some ability of frequent switching among such patterns. This normally involves many components, many different space and time scales, and thus such phenomena like chaos, noise, but, of course, also collectivity and criticality [1]. In fact, due to all those elements, it seems most appropriate to search for a real complexity just at the interface of chaos and collectivity [2,3]. Indeed, these two seemingly contradictory phenomena have

to go in parallel, as they both are connected with existence of many degrees of freedom and a strong, often random, interaction among them.

Approaching complex systems, either empirically or theoretically, is typically based on analyzing large multivariate ensembles of parameters. For this reason, probably the most efficient formal frame to quantify the whole variety of effects connected with complexity is in terms of matrices. Since complexity is embedded in chaos, or even noise, the random matrix theory (RMT) [4,5] provides then an appropriate reference. Its utility results predominantly from the fact that the degree of agreement quantifies the generic properties of a system - those connected with chaotic or noisy activity. For the complex systems this is expected to be a dominant component, but this component is not what constitutes an essence of complexity. From this perspective the deviations are even more relevant and more interesting as they reflect a creative and perhaps deterministic potential emerging from a noisy background of such systems. The main related purpose of the present summary is to identify, within the matrix formalism, some principal characteristics of such deviations - the ones that are common and typical to natural complex dynamical systems.

## 2 Coherence versus noise in matrix representation

Expressed in the most general form, in essentially all the cases of practical interest, the  $n \times n$  matrices  $\mathbf{W}$  used to describe the complex system are by construction designed as

$$\mathbf{W} = \mathbf{X}\mathbf{Y}^T, \quad (1)$$

where  $\mathbf{X}$  and  $\mathbf{Y}$  denote the rectangular  $n \times m$  matrices. Such, for instance, are the correlation matrices whose standard form corresponds to  $\mathbf{Y} = \mathbf{X}$ . In this case one thinks of  $n$  observations or cases, each represented by a  $m$  dimensional row vector  $\mathbf{x}_i$  ( $\mathbf{y}_i$ ), ( $i = 1, \dots, n$ ) [6], and typically  $m$  is larger than  $n$ . In the limit of purely random correlations the matrix  $\mathbf{W}$  is then said to be a Wishart matrix [7]. The resulting density  $\rho_{\mathbf{W}}(\lambda)$  of eigenvalues is here known analytically [8], with the limits ( $\lambda_{min} \leq \lambda \leq \lambda_{max}$ ) prescribed by  $\lambda_{min}^{max} = 1 + 1/Q \pm 2\sqrt{1/Q}$  and  $Q = m/n \geq 1$ . The variance of the elements of  $\mathbf{x}_i$  is here assumed unity.

The more general case, of  $\mathbf{X}$  and  $\mathbf{Y}$  different, results in asymmetric correlation matrices with complex eigenvalues  $\lambda$ . As shown recently [9], such matrices also turn out to provide a very powerful tool in practical applications. In this more general case a limiting distribution corresponding to purely random correlations seems not to be yet known analytically as a function of  $m/n$ . The

result of ref. [10] indicates however that in the case of no correlations, quite generically, one may expect a largely uniform distribution of  $\lambda$  bound in an ellipse on the complex plane.

Further examples of matrices of similar structure, of great interest from the point of view of complexity, include the Hamiltonian matrices of strongly interacting quantum many body systems such as atomic nuclei. This holds true [11] on the level of bound states where the problem is described by the Hermitian matrices, as well as for excitations embedded in the continuum. This latter case can be formulated in terms of an open quantum system [12], which is represented by a complex non-Hermitian Hamiltonian matrix. Several neural network models also belong to this category of matrix structure [13]. In this domain the reference is provided by the Gaussian (orthogonal, unitary, symplectic) ensembles of random matrices with the semi-circle law for the eigenvalue distribution [5]. For the irreversible processes there exists their complex version [14] with a special case, the so-called scattering ensemble [15], which accounts for  $S$ -matrix unitarity.

As it has already been expressed above, several variants of ensembles of the random matrices provide an appropriate and natural reference for quantifying various characteristics of complexity. The bulk of such characteristics is expected to be consistent with RMT, and in fact there exists strong evidence that it is. Once this is established, even more interesting are however deviations, especially those signaling emergence of synchronous or coherent patterns, i.e., the effects connected with the reduction of dimensionality. In the matrix terminology such patterns can thus be associated with a significantly reduced rank  $k$  (thus  $k \ll n$ ) of a leading component of  $\mathbf{W}$ . A satisfactory structure of the matrix that would allow some coexistence of chaos or noise and of collectivity thus reads:

$$\mathbf{W} = \mathbf{W}_r + \mathbf{W}_c. \quad (2)$$

Of course, in the absence of  $\mathbf{W}_r$ , the second term ( $\mathbf{W}_c$ ) of  $\mathbf{W}$  generates  $k$  nonzero eigenvalues, and all the remaining ones ( $n - k$ ) constitute the zero modes. When  $\mathbf{W}_r$  enters as a noise (random like matrix) correction, a trace of the above effect is expected to remain, i.e.,  $k$  large eigenvalues and the bulk composed of  $n - k$  small eigenvalues whose distribution and fluctuations are consistent with an appropriate version of random matrix ensemble. One likely mechanism that may lead to such a segregation of eigenspectra is that  $m$  in eq. (1) is significantly smaller than  $n$ , or that the number of large components makes it effectively small on the level of large entries  $w$  of  $\mathbf{W}$ . Such an effective reduction of  $m$  ( $M = m_{eff}$ ) is then expressed by the following distribution  $P(w)$  of the large off-diagonal matrix elements in the case they

are still generated by the random like processes [11]:

$$P(w) = \frac{|w|^{(M-1)/2} K_{(M-1)/2}(|w|)}{2^{(M-1)/2} \Gamma(M/2) \sqrt{\pi}}, \quad (3)$$

where  $K$  stands for the modified Bessel function. Asymptotically, for large  $w$ , this leads to  $P(w) \sim \exp(-|w|) |w|^{M/2-1}$ , and thus reflects an enhanced probability of appearance of a few large off-diagonal matrix elements as compared to a Gaussian distribution. As consistent with the central limit theorem the distribution (3) quickly converges to a Gaussian with increasing  $M$ .

Another mechanism that may lead to a structure analogous to (2), is the presence of some systematic trend, in addition to noise, in the  $\mathbf{X}$  and  $\mathbf{Y}$  matrices. Then [16], to a first approximation, the second term in this decomposition is represented just by a matrix whose all entries are close in magnitude, and thus its rank is directly seen to be unity. The most straightforward indication that this kind of decomposition applies is an asymmetric shift of  $P(w)$  relative to zero.

Based on several examples of natural complex dynamical systems, like the strongly interacting Fermi systems, the human brain and the financial markets, below we systematize evidence that such effects are indeed common to all the phenomena that intuitively can be qualified as complex.

### 3 Common features of complexity in natural systems

Since it was nuclear physics which gave birth to several concepts relevant to the physics of complex systems, in particular to RMT, we begin with an issue which originates from nuclear considerations and which, at the same time, addresses a problem [17] of great current interest, attracting lot of activity in the literature. More specifically, the related question asks what is a nature of the ground state if the two-body interaction is drawn from a Gaussian ensemble. This is an example of a sparser connectivity than just everything with everything and by this it is more realistic.

In the presence of two-body interactions the many-body Hamiltonian matrix elements  $v_{\alpha,\alpha'}^J$  of good total angular momentum  $J$  in the shell-model basis  $|\alpha\rangle$  generated by the mean field, can be expressed as follows [18]:

$$v_{\alpha,\alpha'}^J = \sum_{J' ii'} c_{J' ii'}^{J\alpha\alpha'} g_{ii'}^{J'}. \quad (4)$$

The summation runs over all combinations of the two-particle states  $|i\rangle$  cou-

pled to the angular momentum  $J'$  and connected by the two-body interaction  $g$ . The analogy of this structure to the one schematically captured by the eq. (2) is evident.  $g_{ii'}^{J'}$  denote here the radial parts of the corresponding two-body matrix elements while  $c_{J'ii'}^{J\alpha\alpha'}$  globally represent elements of the angular momentum recoupling geometry.  $g_{ii'}^{J'}$  are drawn from a Gaussian distribution while the geometry expressed by  $c_{J'ii'}^{J\alpha\alpha'}$  enters explicitly. An explicit calculation then shows [11] that for  $J > 0$  the tails of  $P(v)$  are very nicely reproduced by the eq. (3) with  $M \approx 2$ . This originates from the fact that a quasi-random coupling of individual spins results in the so-called geometric chaoticity [19] and thus  $c_{J'ii'}^{J\alpha\alpha'}$  coefficients are also Gaussian distributed. In this case, these two ( $g$  and  $c$ ) essentially random ingredients lead however to an order of magnitude larger separation of the ground state from the remaining states as compared to a pure RMT limit, and this is consistent with the above estimate for  $M$ . Due to more severe selection rules the effect of geometric chaoticity does not apply for  $J = 0$ . As a consequence, in this particular case  $P(v)$  is much closer to a Gaussian, i.e.,  $M$  is here much larger. Consistently, the ground state energy gaps measured relative to the average level spacing characteristic for a given  $J$  is larger for  $J > 0$  than for  $J = 0$ , and also  $J > 0$  ground states are more orderly than those for  $J = 0$ , as it can be quantified in terms of the information entropy [16], for instance.

Interestingly, such reductions of dimensionality of the Hamiltonian matrix can also be seen [20] locally in explicit calculations with realistic (non-random) nuclear interactions. A collective state, the one which turns out coherent with some operator representing physical external field, is always surrounded by a reduced density of states, i.e., it repels the other states. It is also appropriate to mention at this point that similar effects of reduced dimensionality, applicability of the formula (3), and of the resulting segregation of states one observes [21–23] in the many body quantum open systems due to the coupling to continuum. Of course, in the latter case on the complex plane. In all those cases, the global fluctuation characteristics remain however largely consistent with the corresponding version of the random matrix ensemble.

Recently, a broad arena of applicability of the random matrix theory opens in connection with the most complex systems known to exist in the universe. With no doubt, the most complex is the human's brain and those phenomena that result from its activity. From the physics point of view the financial world, reflecting such an activity, is of particular interest [24] because its characteristics are quantified directly in terms of numbers and a huge amount of electronically stored financial data is readily available. An access to a single brain activity is also possible by detecting the electric or magnetic fields generated by the neuronal currents [25]. With the present day techniques of electro- or magnetoencephalography, in this way it is possible to generate the time series which resolve neuronal activity down to the scale of 1 ms.

One may debate over what is more complex, the human brain or the financial world, and there is no unique answer. It seems however to us that it is the financial world that is even more complex. After all, it involves the activity of many human brains and it seems even less predictable due to more frequent changes between different modes of action. Noise is of course overwhelming in either of these systems, as it can be inferred from the structure of eigen-spectra of the correlation matrices taken across different space areas at the same time [26,27,16], or across different time intervals [9,28]. There however always exist several well identifiable deviations, which, with help of reference to the universal characteristics of the random matrix theory, and with the methodology briefly reviewed above, can be classified as real correlations or collectivity. An easily identifiable gap between the corresponding eigenvalues of the correlation matrix and the bulk of its eigenspectrum plays the central role in this connection. The brain when responding to the sensory stimulations develops larger gaps than the brain at rest [9]. The correlation matrix formalism in its most general asymmetric form allows to study [9] also the time-delayed correlations, like the ones between the opposite hemispheres. The time-delay reflecting the maximum of correlation (time needed for an information to be transmitted between the different sensory areas in the brain [29,30]) is also associated with appearance of one significantly larger eigenvalue. Similar effects appear to govern formation of the heteropolymeric biomolecules. The ones that nature makes use of are separated by an energy gap from the purely random sequences [31].

As far as the dynamics of evolution of complex systems is concerned one interesting observation made in [16], based on the stock market evolution, is also to be pointed out in the present context. It appears that increases, as a rule, are more competitive, and thus less collective than decreases which are always accompanied by a more violent collectivity. This may illustrate a more general logic of evolution of natural complex dynamical systems.

Such characteristics of coherence are typically connected with a few largest eigenvalues of the correlation matrix and those eigenvalues stay significantly above  $\lambda_{max}$ . The bulk of eigenvalues is quite universally consistent with the RMT limit. There exist however some more subtle measures of eigenvalue fluctuations in the random matrix theory. In particular, we refer here to the Tracy-Widom law [32] which, based on Painlevé representations and a proper scaling of eigenvalues, provides a general formalism to study fluctuations of individual eigenvalues. An appropriate rescaling of eigenvalues makes this law applicable also to Wishart matrices [33]. Motivated by this formalism, and based on the high frequency recordings of all the stocks comprised by DAX, we recently analysed the fluctuations of various eigenvalues of the correlation matrix using our methodology [16] of moving time-window. Even though, on average, the distribution of the bulk of eigenvalues is here consistent with the RMT limit, significant deviations relative to fluctuations of eigenvalues

of the correlation matrices calculated from purely random time series still remain. This can partly be attributed to the effect reminiscent of the slaving principle of synergetics [34]: if one, or a small fraction of states take the entire collectivity, all the others become enslaved and thus their 'noise freedom' gets also reduced. This may affect the oposite edge of the spectrum by suppressing the amount of noise there, and thus making the corresponding states again deviating more from their RMT limit. The above seems to provide a likely explanation for the effect observed in ref. [27], that the smallest eigenvalues of the financial correlation matrix also correspond to more localised states. Such effects seem to constitute another manifestation of complexity.

## 4 Summary

The above brief review tempts to view complexity as a trinity comprising coherence, chaos and a gap (probably not too large) between them. Coherence constitutes the essence as it makes patterns and structures, which is of primary interest and importance. Chaos is always present in any really interesting system and, in fact, it is even needed as it allows to quickly explore the whole available phase space, and thus to probe various possibilities and to switch from one pattern of activity to another. Finally, the gap between them allows the structures to be identifiable and to exist for some time. Thus all the three are needed in parallel. Such a combination probably makes a natural system most efficient in its evolution.

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